TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 267, Number 1, September 1981

ALGEBRAIC EXTENSIONS OF POWER SERIES RINGS

BY

JIMMY T. ARNOLD

ABSTRACT. Let D and J be integral domains such that $D \subset J$ and $J[[X]]$ is not algebraic over $D[[X]]$. Is it necessarily the case that there exists an integral domain R such that $D[[X]] \subset R \subseteq J[[X]]$ and $R \cong D[[X]][[\{Y_i\}_{i=1}^{\infty}]]$? While the general question remains open, the question is answered affirmatively in a number of cases. For example, if D satisfies any one of the conditions (1) D is Noetherian, (2) D is integrally closed, (3) the quotient field K of D is countably generated as a ring over D, or (4) D has Krull dimension one, then an affirmative answer is given. Further, in the Noetherian case it is shown that $J[[X]]$ is algebraic over $D[[X]]$ if and only if it is integral over $D[[X]]$ and necessary and sufficient conditions are given on D and J in order that this occur. Finally if, for every positive integer n, $D[[X_1, \ldots, X_n]] \subset R \subseteq J[[X_1, \ldots, X_n]]$ implies that $R \ncong D[[X_1, \ldots, X_n]][[\{Y_i\}_{i=1}^{\infty}]]$, then it is shown that $J[[X_1, \ldots, X_n]]$ is algebraic over $D[[X_1, \ldots, X_n]]$ for every n.

1. Introduction. Throughout this paper D and J denote integral domains (with identity) having quotient fields K and L, respectively, and such that $D \subseteq J$. If $L = K$ we say that J is an *overring* of D. In the case when $J = K$ Gilmer has shown in [2] that $J[[X]]$ is an overring of $D[[X]]$ if and only if $\bigcap_{i=1}^{\infty}(a_i) \neq (0)$ for each subset $\{a_i\}_{i=1}^{\infty}$ of nonzero elements of D. Sheldon has shown in [5] that if a is a nonzero element of D such that $\bigcap_{i=1}^{\infty}(a^i) = (0)$ and if $J = D[1/a]$, then the quotient field of $J[[X]]$ has infinite transcendence degree over the quotient field of $D[[X]]$ (hereafter we will call this the transcendence degree of $J[[X]]$ over $D[[X]]$). Thus, if $J = D[1/a]$ then either $J[[X]]$ is an overring of $D[[X]]$ or has infinite transcendence degree over $D[[X]]$. Motivated by these results, Arnold and Boyd asked in [1] whether this is the case for an arbitrary overring J of D. A simple example is given in [1, Example 1.2] of a domain D with characteristic $p \neq 0$ and an overring J of D such that $(J[[X]])^p \subseteq D[[X]]$ but $J[[X]]$ is not an overring of $D[[X]]$. Thus, if D has characteristic $p \neq 0$ the appropriate question is whether either $J[[X]]$ has infinite transcendence degree over $D[[X]]$ or there exists a nonnegative integer m such that $(J[[X]])^{p^m}$ is contained in the quotient field of $D[[X]]$. This led, in [1], to the consideration of the equivalence of statements (a)–(d) below. Statement (e) is included for future reference.

(1.1) *In each of the following statements J is an overring of D, $D^* = D \setminus (0)$, and p is the characteristic of D if D has nonzero characteristic, while $p = 1$ if D has characteristic zero.*

Received by the editors March 7, 1980 and, in revised form, June 24, 1980.
AMS (MOS) subject classifications (1970). Primary 13J05; Secondary 13G05, 13B99.
Key words and phrases. Power series ring, algebraic extension, integral extension.

0002-9947/81/0000-0406/$05.00

(a) *For each subset* $\{\xi_i\}_{i=1}^{\infty}$ *of* J *there exists a nonnegative integer* m *and an element* d *in* D^* *such that* $d\xi_i^{p^m} \in D$ *for each* i.

(b) *There exists a nonnegative integer* m *such that* $(J[[X]])^{p^m} \subseteq (D[[X]])_{D^*}$.

(c) $J[[X]]$ *is algebraic over* $D[[X]]$.

(d) $J[[X]]$ *has finite transcendence degree over* $D[[X]]$.

(e) *If* R *is an integral domain such that* $D[[X]] \subset R \subseteq J[[X]]$, *then* $R \not\cong D[[X]][[\{Y_i\}_{i=1}^{\infty}]]$ *via a* $D[[X]]$-*isomorphism*.

In general the implications (a) $\leftrightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (d) hold [**1**, Proposition 2.1] and it is shown in [**1**] that conditions (a)–(d) are equivalent if D is Noetherian [**1**, Theorem 2.5], if D is root closed [**1**, Theorem 1.6], if K is countably generated as a ring over D [**1**, Theorem 2.4], or if J is a quotient overring of D [**1**, Theorem 1.10]. Whether (a)–(d) are equivalent in general is an open question. In this paper we wish to remove the assumption that J is an overring of D and, since clearly (d) $\rightarrow$ (e), we wish to strengthen the basic question asked in [**1**] to the following. Is it the case that either $J[[X]]$ is algebraic over $D[[X]]$ (in some "nice" way) or there exists an integral domain R such that $D[[X]] \subset R \subseteq J[[X]]$ and $R \cong D[[X]][[\{Y_i\}_{i=1}^{\infty}]]$? To be more specific, we ask whether statements (1)–(5) below are equivalent.

(1.2) *In each of the following statements* $D^* = D \setminus (0)$, K_0 *is the maximal, separable extension of* K *in* L, *and* p *is the characteristic of* D *if* D *has nonzero characteristic while* $p = 1$ *if* D *has zero characteristic.*

(1) L *is algebraic over* K, $[K_0: K]$ *is finite, and if* D_0 *is any integral domain such that* $D \subseteq D_0 \subseteq J$ *and* D_0 *has quotient field containing* K_0, *then for each subset* $\{\xi_i\}_{i=1}^{\infty}$ *of* J *there exists a nonnegative integer* m *and a nonzero element* d *in* D *such that* $d\xi_i^{p^m} \in D_0$ *for each positive integer* i.

(2) *There exists a nonnegative integer* m *and a finite integral extension* $D_0 = D[\theta]$ *of* D *such that* $K_0 = K(\theta)$, D_0 *has quotient field* K_0, $D_0 \subseteq J$, *and* $(J[[X]])^{p^m} \subseteq (D_0[[X]])_{D^*}$. *If* $K_0 = K$ *we can take* $D = D_0$.

(3) $J[[X]]$ *is algebraic over* $D[[X]]$.

(4) $J[[X]]$ *has finite transcendence degree over* $D[[X]]$.

(5) *If* R *is an integral domain such that* $D[[X]] \subset R \subseteq J[[X]]$ *then* $R \not\cong D[[X]][[\{Y_i\}_{i=1}^{\infty}]]$ *via a* $D[[X]]$-*isomorphism*.

§2 is devoted primarily to the proof of the key result of this paper, Theorem 2.1, which gives necessary conditions in order that (5) of (1.2) hold. In §3 it is shown that (1) $\leftrightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (5) and sufficient conditions are given on D and J in order that (1)–(5) be equivalent. In particular, (1)–(5) are shown to be equivalent if D is Noetherian (Theorem 3.9), if D is integrally closed (Theorem 3.10), if the quotient field K of D is countably generated as a ring over D (Theorem 3.11), if D has Krull dimension one (Theorem 3.12), or if J is a quotient overring of some finite integral extension of D (Theorem 3.14). In [**3**] Gilmer obtains necessary and sufficient conditions on the fields K and L, with $K \subseteq L$, in order that $L[[X]]$ be integral over $K[[X]]$. Theorem 3.9 shows that if D is Noetherian, then $J[[X]]$ is algebraic over $D[[X]]$ if and only if it is integral over $D[[X]]$ and the necessary and sufficient conditions established by Gilmer in [**3**] are extended to this case. The basic question remains open.

(1.3) *Open question. Are conditions* (1)–(5) *of* (1.2) *equivalent*?

We conclude §3 by proving, in Theorem 3.17, that $J[[X_1, \ldots, X_n]]$ is algebraic over $D[[X_1, \ldots, X_n]]$ for every positive integer n if and only if, for every n, $D[[X_1, \ldots, X_n]] \subset R \subseteq J[[X_1, \ldots, X_n]]$ implies that

$$R \ncong D[[X_1, \ldots, X_n]][[\{Y_i\}_{i=1}^{\infty}]].$$

Throughout the paper $D^* = D \setminus (0)$, K_0 denotes the maximal, separable extension of K in L [**6**, p. 123], and p denotes the characteristic of D if D has nonzero characteristic while $p = 1$ if D has characteristic zero. If L/K is algebraic then L/K_0 is purely inseparable and we say that L *has finite exponent over* K_0 if there exists a nonnegative integer n such that $L^{p^n} \subseteq K_0$ [**6**, p. 123].

For a commutative ring R, $R[[X]]$ denotes the power series ring over R. If $f = \sum_{i=0}^{\infty} r_i X^i \in R[[X]]$ then the *order* of f, denoted $o(f)$, is the smallest integer n such that $r_n \neq 0$. If I is an indexing set we write Y_I (or Y if I is understood) to denote the set $\{Y_i\}_{i \in I}$ of analytic indeterminates over R and $R[[Y_I]]$ denotes the full power series ring in these indeterminates. (In the notation of [**4**, p. 10], $R[[Y_I]]$ denotes the ring $R[[\{Y_i\}_{i \in I}]]_3$.) If $\Lambda(I) = \bigoplus_{i \in I} S_i$, where each S_i is the additive semigroup of nonnegative integers, and if π_i: $\Lambda(I) \to S_i$ is the canonical projection, then for $\alpha \in \Lambda(I)$ we set $Y_I^{\alpha} = \prod_{\{i \in I | \pi_i(\alpha) \neq 0\}} Y_i^{\pi_i(\alpha)}$ and we write $f = \sum_{\alpha \in \Lambda(I)} r_\alpha Y^\alpha$ to denote an arbitrary element of $R[[Y_I]]$. We are interested only in the case when I is countable. In this case suppose that $D \subset J$ and let $\mathfrak{F} = \{f_i\}_{i \in I}$ be a subset of $XJ[[X]]$ such that $\{f_i \in \mathfrak{F} | o(f_i) \leqslant k\}$ is finite for each positive integer k. Then the correspondence $Y_i \to f_i$ determines a unique $D[[X]]$-homomorphism ϕ: $D[[X]][[Y_I]] \to J[[X]]$ and we write $D[[X]][[\mathfrak{F}]]$ to denote the image of ϕ. We use ω to denote the set of positive integers. Thus, in (5) of (1.2) we have $D[[X]][[\{Y_i\}_{i=1}^{\infty}]] = D[[X]][[Y_\omega]]$.

2. Main theorem. In §3 we will give several conditions under which (1)–(5) of (1.2) are equivalent. In view of Proposition 3.7 it will suffice to show that (5) $\to$ (1) and our first result, Theorem 2.1, is the key result in this direction, for it gives necessary conditions in order that (5) of (1.2) hold for the domains $D \subset J$. For later convenience we actually state and prove Theorem 2.1 in the more general setting in which the domains $A \subset B$ satisfy (5) of (1.2), $A \subseteq D$ and $J = D[B]$. In the statement and proof of Theorem 2.1 $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of positive integers defined as follows. First let $\{\nu_{ij} | i \geqslant 1, j \geqslant i\}$ be a set of positive integers such that $\nu_{1,1} > 1$, $\nu_{1,k} > k(1 + \sum_{j<k, i \leqslant j} \nu_{ij})$ for $k > 1$, and $\nu_{ik} = \sum_{j=i-1}^{k} \nu_{i-1,j}$ for $i > 1$. For each $k \geqslant 1$ we set $\gamma_k = \nu_{kk}$.

THEOREM 2.1. *Let A and B be integral domains such that $A \subset B$ and for each integral domain R such that $A[[X]] \subset R \subseteq B[[X]]$ assume that $R \ncong A[[X]][[Y_\omega]]$ via an $A[[X]]$-isomorphism. Let D and J be integral domains with quotient fields K and L, respectively, and suppose that $A \subseteq D \subseteq J$ and $J = D[B]$. If K_0 is the maximal separable extension of K in L then L is algebraic over K, L/K_0 is purely inseparable with finite exponent, and $[K_0: K]$ is finite. Further, if D_0 is any integral domain with quotient field containing K_0 and such that $D \subseteq D_0 \subseteq J$ then for each subset $\{\xi_i\}_{i=1}^{\infty}$ of*

B there exist integers $n \geqslant 1$ and $m \geqslant 0$ and a nonzero element d in D such that for each $k \geqslant 1$, $d\xi_k^{p^m}$ is a polynomial in $D_0[\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}]$ with total degree not more than $p^m\gamma_{k+n}$ in $\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}$.

In proving Theorem 2.1 we let $\{\xi_i\}_{i=1}^\infty$ be a subset of B and define a set $\mathfrak{F} = \{f_i\}_{i=1}^\infty \subseteq A[\{\xi_i\}_{i=1}^\infty][[X]]$ such that $o(f_i) \geqslant i$. By assumption the homomorphism ϕ: $A[[X]][[Y_\omega]] \to A[[X]][[\mathfrak{F}]]$ defined by $\phi(Y_i) = f_i$ is not an isomorphism. Therefore, there exists a nonzero power series $G(Y)$ in $A[[X]][[Y_\omega]]$ such that $G(\mathfrak{F}) = 0$. In $G(\mathfrak{F})$ we wish to examine the coefficients of selected powers of X and thus obtain polynomial equations in the ξ_i. In order to simplify the description of the desired coefficients we first reduce, via Lemma 2.2 and Corollary 2.3, to the case in which we have a single element $f \in A[\{\xi_i\}_{i=1}^\infty][[X]]$ and a nonzero power series $H(Y)$ in $A_1[[Y]]$, where A_1 is an appropriate extension of $A[[X]]$, such that $H(f) = 0$.

In order that we may simultaneously consider the cases $p = 1$ and $p > 1$, in our next result, Lemma 2.2, we consider a situation somewhat more general than that described above. Namely, for each positive integer i, let $\mathfrak{F}_i$ be a countable (perhaps finite) subset of $XB[[X]]$ indexed by the set Δ_i. Set $\mathfrak{F} = \bigcup_{i=1}^\infty \mathfrak{F}_i$, $\Delta = \bigcup_{i=1}^\infty \Delta_i$ and for each positive integer k let $\mathfrak{F}_{(k)} = \bigcup_{i=1}^k \mathfrak{F}_i$, $\mathfrak{F}^{(k)} = \bigcup_{i=k+1}^\infty \mathfrak{F}_i$, $\Delta_{(k)} = \bigcup_{i=1}^k \Delta_i$, and $\Delta^{(k)} = \bigcup_{i=k+1}^\infty \Delta_i$. We further assume that $\{f \in \mathfrak{F} \mid o(f) < k\}$ is finite for each positive integer k.

LEMMA 2.2. *Let A_1 be an integral domain such that $A[[X]] \subseteq A_1 \subseteq B[[X]]$ and let $G(Y) = \sum_{\alpha \in \Lambda(\Delta)} a_\alpha Y^\alpha$ be a nonzero element of $A_1[[Y_\Delta]]$ such that $G(\mathfrak{F}) = 0$. Then there exists an integer t and a nonzero power series $H(Y) = \sum_{\beta \in \Lambda(\Delta_t)} b_\beta Y^\beta$ in $A_1[[\mathfrak{F}^{(t)}]][[Y_{\Delta_t}]]$ such that $H(\mathfrak{F}_t) = 0$. Further, if $p > 1$ and $\pi_\lambda(\alpha) < p$ for each $\lambda \in \Delta$ and for each $\alpha \in \Lambda(\Delta)$ such that $a_\alpha \neq 0$, then $\pi_\lambda(\beta) < p$ for each $\lambda \in \Delta_t$ and for each $\beta \in \Lambda(\Delta_t)$ such that $b_\beta \neq 0$.*

PROOF. Choose α_0 so that $o(a_{\alpha_0}) = \min\{o(a_\alpha) \mid \alpha \in \Lambda(\Delta)\}$. Since $G(\mathfrak{F}) = 0$ we have $o(a_0) = o(-\sum_{\alpha \in \Lambda(\Delta)\setminus(0)} a_\alpha \mathfrak{F}^\alpha)$. But $o(f) \geqslant 1$ for each $f \in \mathfrak{F}$ so it follows that $\alpha_0 \neq 0$. Let s be a positive integer such that $\pi_\lambda(\alpha_0) = 0$ for each $\lambda \in \Delta^{(s)}$ and write $G(Y) = \sum_{\gamma \in \Lambda(\Delta_{(s)})} g_\gamma Y^\gamma \in A_1[[Y_{\Delta^{(s)}}]][[Y_{\Delta_{(s)}}]]$. Let $\gamma_0 \in \Lambda(\Delta_{(s)})$ be such that $\pi_\lambda(\gamma_0) = \pi_\lambda(\alpha_0)$ for each $\lambda \in \Delta_{(s)}$. If we write $g_{\gamma_0} = \sum_{\eta \in \Lambda(\Delta^{(s)})} d_\eta Y^\eta \in A_1[[Y_{\Delta^{(s)}}]]$ then $d_0 = a_{\alpha_0}$ and for $\eta \in \Lambda(\Delta^{(s)})$ each nonzero d_η is one of the coefficients a_α of $G(Y)$. If $g_{\gamma_0}(\mathfrak{F}^{(s)}) = 0$ then $d_0 = -\sum_{\eta \in \Lambda(\Delta^{(s)})\setminus(0)} d_\eta(\mathfrak{F}^{(s)})^\eta$ and since $o(f) \geqslant 1$ for $f \in \mathfrak{F}^{(s)}$ this contradicts that $o(d_0) \leqslant o(d_\eta)$ for each nonzero d_η. Therefore $g_{\gamma_0}(\mathfrak{F}^{(s)}) \neq 0$ so if we set $c_\gamma = g_\gamma(\mathfrak{F}^{(s)})$ for each $\gamma \in \Lambda(\Delta_{(s)})$ then we obtain a nonzero power series $G_1(Y) = \sum_{\gamma \in \Lambda(\Delta_{(s)})} c_\gamma Y^\gamma \in A_1[[\mathfrak{F}^{(s)}]][[Y_{\Delta_{(s)}}]]$ such that $G_1(\mathfrak{F}_{(s)}) = 0$. Clearly, if $p > 1$ and $\pi_\lambda(\alpha) < p$ for each $\lambda \in \Delta$ and for each $\alpha \in \Lambda(\Delta)$ such that $a_\alpha \neq 0$, then $\pi_\lambda(\gamma) < p$ for each $\lambda \in \Delta_{(s)}$ and for each $\gamma \in \Lambda(\Delta_{(s)})$ such that $c_\gamma \neq 0$.

The existence of G_1 allows us to choose a minimal subset $\{t_1, \ldots, t_r\}$ of $\{1, \ldots, s\}$ for which there exists a nonzero power series $H_1(Y) = \sum_{\gamma \in \Lambda(\Delta_{t_1} \cup \cdots \cup \Delta_{t_r})} c_\gamma Y^\gamma$ in $A_1[[\mathfrak{F}^{(s)}]][[Y_{\Delta_{t_1} \cup \cdots \cup \Delta_{t_r}}]]$ with $H_1(\mathfrak{F}_{t_1}, \ldots, \mathfrak{F}_{t_r}) = 0$. Further, we assume $H_1(Y)$ is chosen so that if $p > 1$ then $\pi_\lambda(\gamma) < p$ for each $\lambda \in \Delta_{t_1} \cup \cdots \cup \Delta_{t_r}$ and for each $\gamma \in \Lambda(\Delta_{t_1} \cup \cdots \cup \Delta_{t_r})$ such that $c_\gamma \neq 0$. If we write $H_1(Y) = \sum_{\beta \in \Lambda(\Delta_{t_1})} h_\beta Y^\beta \in A_1[[\mathfrak{F}^{(s)}, Y_{\Delta_{t_2} \cup \cdots \cup \Delta_{t_r}}]][[Y_{\Delta_{t_1}}]]$ then $h_\beta \neq 0$ for some $\beta \in \Lambda(\Delta_{t_1})$ and by choice of $\{t_1, \ldots, t_r\}$, $h_\beta(\mathfrak{F}_{t_2}, \ldots, \mathfrak{F}_{t_r}) \neq 0$. Thus, if $b_\beta = h_\beta(\mathfrak{F}_{t_2}, \ldots, \mathfrak{F}_{t_r})$ for each $\beta \in \Lambda(\Delta_{t_1})$ then $H(Y) = \sum_{\beta \in \Lambda(\Delta_{t_1})} b_\beta Y^\beta \in A_1[[\mathfrak{F}^{(s)}, \mathfrak{F}_{t_2}, \ldots, \mathfrak{F}_{t_r}]][[Y_{\Delta_{t_1}}]] \subseteq A_1[[\mathfrak{F}^{(t_1)}]][[Y_{\Delta_{t_1}}]]$ is a nonzero power series such that $H(\mathfrak{F}_{t_1}) = 0$. Again it is clear that if $p > 1$ then $\pi_\lambda(\beta) < p$ for each $\lambda \in \Delta_{t_1}$ and for each $\beta \in \Lambda(\Delta_{t_1})$ such that $b_\beta \neq 0$.

COROLLARY 2.3. *Let $\mathfrak{F} = \{f_i\}_{i=1}^\infty$ be a subset of $XB[[X]]$ such that $o(f_i) \geqslant i$ for each i. If there exists a nonzero element $G(Y)$ in $A[[X]][[Y_\omega]]$ such that $G(\mathfrak{F}) = 0$ then there exist integers $s \geqslant 0$ and $t \geqslant 1$ and a nonzero element $H(Y)$ in $A[[X, \{f_i^{p^j} \mid j = s$ and $i > t$ or $p > 1$ and $j > s\}]][[Y]]$ such that $H(f_t^{p^s}) = 0$. Further, if $p > 1$ then $H(Y)$ is a polynomial with degree less than p.*

PROOF. We first consider the case $p = 1$. Thus, we wish to establish the existence of an integer t and a nonzero element $H(Y)$ in $A[[X, \{f_i\}_{i=t+1}^\infty]][[Y]]$ such that $H(f_t) = 0$. This is an immediate consequence of Lemma 2.2 with $\mathfrak{F}_i = \{f_i\}$ for each $i \geqslant 1$.

Now assume that $p > 1$ and suppose the notation is such that $0 = G(\mathfrak{F}) = \phi(G(Y))$ where ϕ: $A[[X]][[Y_\omega]] \to A[[X]][[\mathfrak{F}]]$ is the homomorphism such that $\phi(Y_i) = f_i$ for each $i \in \omega$. For each positive integer n we can, by the division algorithm, write n uniquely in the form $n = n_0 + n_1 p + \cdots + n_k p^k$ where $0 \leqslant n_i < p$. Thus $Y_i^n = Y_i^{n_0}(Y_i^p)^{n_1} \cdots (Y_i^{p^k})^{n_k}$ and we define a nonzero element $G_1(Y)$ in $A[[X]][[Y_{\omega \times \omega}]]$ by replacing Y_i^n in $G(Y)$ with $Y_{i,1}^{n_0} \cdots Y_{i,k+1}^{n_k}$. For each positive integer j set $\mathfrak{F}_j = \{f_i^{p^{j-1}}\}_{i=1}^\infty$ and, changing the notation established in the corollary, let $\mathfrak{F} = \bigcup_{j=1}^\infty \mathfrak{F}_j$. If ϕ_1: $A[[X]][[Y_{\omega \times \omega}]] \to A[[X]][[\mathfrak{F}]]$ is defined by $\phi_1(Y_{ij}) = f_i^{p^{j-1}}$ then $\phi(G) = \phi_1(G_1) = G_1(\mathfrak{F}) = 0$. Further, if $G_1(Y) = \sum_{\alpha \in \Lambda(\omega \times \omega)} a_\alpha Y_{\omega \times \omega}^\alpha$ then $\pi_\lambda(\alpha) < p$ for each $\lambda \in \omega \times \omega$ and for each $\alpha \in \Lambda(\omega \times \omega)$ such that $a_\alpha \neq 0$. By Lemma 2.2 there exists an integer $s \geqslant 0$ and a nonzero power series $H_1(Y) = \sum_{\beta \in \Lambda(\omega)} b_\beta Y^\beta$ in $A[[X, \{f_i^{p^j}\}_{j=s+1}^\infty]][[Y_\omega]]$ such that $H_1(\mathfrak{F}_{s+1}) = 0$ and such that $\pi_\lambda(\beta) < p$ for each $\lambda \in \omega$ and each $\beta \in \Lambda(\omega)$ such that $b_\beta \neq 0$. Set $A_1 = A[[X, \{f_i^{p^j}\}_{j=s+1}^\infty]]$, for each positive integer i let $\mathfrak{F}_i' = \{f_i^{p^s}\}$, and set $\mathfrak{F}' = \bigcup_{i=1}^\infty \mathfrak{F}_i' = \{f_i^{p^s}\}_{i=1}^\infty$. By Lemma 2.2 there exists a positive integer t and a nonzero element $H(Y) = \sum_{i=0}^\infty c_i Y^i$ in $A_1[[\{f_i^{p^s}\}_{i=t+1}^\infty]][[Y]]$ such that $H(f_t^{p^s}) = 0$ and $i < p$ if $c_i \neq 0$. This completes the proof of Corollary 2.3.

In our next two results, Lemma 2.5 and Corollary 2.7, we examine the coefficients of selected powers of X in an expression of the form $H(f) = 0$. Specifically, let $H(Y) = \sum_{i=0}^\infty a_i(X) Y^i$ be in $A_1[[Y]]$ where $A[[X]] \subseteq A_1 \subseteq B[[X]]$ and where $a_i(X) = \sum_{j=0}^\infty y_{ij} X^j$ for each $i \geqslant 0$. Set $f = \sum_{i=t}^\infty \zeta_i X^{\nu_i}$ where $\{\zeta_i\}_{i=t}^\infty$ is a subset of B and where $\{\nu_i\}_{i=t}^\infty$ is a sequence of positive integers such that $\nu_t > t \geqslant 1$ and $\nu_k > k(1 + \sum_{i=t}^{k-1} \nu_i)$ for $k > t$. For a positive integer n let $k_1 < \cdots < k_n$ be a sequence of integers such that $k_1 > t$ and for $1 \leqslant j \leqslant n$ set $\mu_j = \nu_{k_1} + \cdots + \nu_{k_j}$. In Lemma 2.5 we consider the coefficient of X^{μ_n} in $H(f)$.

If $g(X) = \sum_{i=0}^\infty b_i X^{\lambda_i}$ and if α is a nonnegative integer then $g_{(\alpha)}$ denotes the truncation $\sum_{i=0}^\alpha b_i X^{\lambda_i}$ of g. The following observation is easily proved by induction on k (cf. [**1**, 3.2]) and will be useful in the proof of Lemma 2.5.

(2.4) *If α and β are integers with $t \leqslant \alpha < \beta$ then for each positive integer k we can write*

$$f_{(\beta)}^k = f_{(\alpha)}^k + g + k\zeta_\beta X^{\nu_\beta} f_{(\alpha)}^{k-1} + h$$

where $o(h) \geqslant \nu_\beta + \nu_{\alpha+1}$, $g = 0$ if $\alpha = \beta - 1$, $o(g) \geqslant \nu_{\alpha+1}$ if $\alpha < \beta - 1$, and $g \in D[\zeta_t, \ldots, \zeta_{\beta-1}][X]$.

We note that Lemma 2.5 is a slightly altered version of Lemma 3.1 of **[1]** and our proof parallels that given in **[1]**.

LEMMA 2.5. *With notation as above let $\{k_i\}_{i=1}^n$ be a fixed set of positive integers such that $t < k_1 < \cdots < k_n$ and set $\mu_n = \nu_{k_1} + \cdots + \nu_{k_n}$. If ϕ is the coefficient of X^{μ_n} in $H(f)$ then ϕ determines polynomials $\{\phi_i\}_{i=0}^n$ and $\{\psi_i\}_{i=0}^{n-1}$ such that $\phi = \phi_n$ and the following conditions hold*:

(1) *For $0 \leqslant m < n$, $\phi_{m+1} = \phi_m \zeta_{k_{m+1}} + \psi_m$.*

(2) *For $0 \leqslant m < n$, ϕ_m is a polynomial in $A[\{y_{ij} | 0 \leqslant i \leqslant \mu_n,\ 0 \leqslant j \leqslant \mu_m\}, \zeta_t, \ldots, \zeta_{k_m}]$ and is independent of the choice of $k_{m+1}, \ldots, k_n$.*

(3) *For $0 \leqslant m < n$, ψ_m is a polynomial in $A[\{y_{ij} | 0 \leqslant i \leqslant \mu_n,\ 0 \leqslant j \leqslant \mu_{m+1}\}, \zeta_t, \ldots, \zeta_{k_{m+1}-1}]$ which has total degree one in the y_{ij} and total degree not greater than μ_n in the ζ_i.*

(4) *$\phi_0 = n! y_{n,0}$.*

PROOF. Let ϕ_{sn} denote the coefficient of X^{μ_n} in $a_s f^s$. If $s > \mu_n$ then $o(f^s) > \mu_n$ so $\phi_{sn} = 0$. Thus, we assume that $0 \leqslant s \leqslant \mu_n$. Clearly the coefficient of X^{μ_n} in $H(f)$ is $\phi_n = \sum_{s=0}^{\mu_n} \phi_{sn}$ so the lemma follows if we prove that each ϕ_{sn} determines polynomials $\{\phi_{si}\}_{i=0}^n$ and $\{\psi_{si}\}_{i=0}^{n-1}$ that satisfy conditions (1)–(3) and have the property that $\phi_{s,0} = 0$ if $s \neq n$ while $\phi_{n,0} = n! y_{n,0}$. For the remainder of the proof we are, therefore, interested only in the coefficient of X^{μ_n} in $a_s f^s$. Thus, we fix s and drop it from the notation ϕ_{si} and ψ_{si}. The following result will simplify the process of describing the polynomials $\{\phi_i\}_{i=0}^n$ and $\{\psi_i\}_{i=0}^{n-1}$.

(2.6) *For $s < n$ and $n - s \leqslant m \leqslant n$ and for $s \geqslant n$ and $1 \leqslant m \leqslant n$ set $P_m(X) = (a_s)_{(\mu_m)}[(s!/(s-n+m)!) f_{(k_m)}^{s-n+m}]$. If $m < n$ then $P_{m+1}(X) = Q_m(X) + P_m(X)\zeta_{k_{m+1}} X^{\nu_{k_{m+1}}} + H_m(X)$ where $o(H_m) > \mu_{m+1}$ and $Q_m(X)$ has coefficients in $A[\{y_{sj}\}_{j=0}^{\mu_{m+1}}, \zeta_t, \ldots, \zeta_{k_{m+1}-1}]$ with total degree one in the y_{sj} and total degree not greater than μ_n in the ζ_i.*

To prove (2.6) suppose that $m < n$. By assumption we have $P_{m+1}(X) = (a_s)_{(\mu_{m+1})}[(s!/(s-n+m+1)!) f_{(k_{m+1})}^{s-n+m+1}]$ and by (2.4) we can write $f_{(k_{m+1})}^{s-n+m+1} = f_{(k_m)}^{s-n+m+1} + g + (s-n+m+1)\zeta_{k_{m+1}} X^{\nu_{k_{m+1}}} f_{(k_m)}^{s-n+m} + h$ where $o(h) \geqslant \nu_{k_{m+1}} + \nu_{k_m+1}$, $o(g) \geqslant \nu_{k_m+1}$, and $g \in D[\zeta_t, \ldots, \zeta_{k_{m+1}-1}][X]$. Making this substitution for $f_{(k_{m+1})}^{s-n+m+1}$ in $P_{m+1}(X)$ we may write $P_{m+1}(X) = Q_m(X) + P_m(X)\zeta_{k_{m+1}} X^{\nu_{k_{m+1}}} + H_m(X)$ where $Q_m(X) = (a_s)_{(\mu_{m+1})}[(s!/(s-n+m+1)!)(f_{(k_m)}^{s-n+m+1} + g)]$ and

$$H_m(X) = \left(\sum_{j=\mu_m+1}^{\mu_{m+1}} y_{sj} X^j\right)\left[(s!/(s-n+m)!)\zeta_{k_{m+1}} X^{\nu_{k_{m+1}}} f_{(k_m)}^{s-n+m}\right]$$
$$+ (a_s)_{(\mu_{m+1})}\left[(s!/(s-n+m+1)!)h\right].$$

Now $o(h) \geqslant \nu_{k_{m+1}} + \nu_{k_m+1} > \nu_{k_{m+1}} + \sum_{i=t}^{k_m} \nu_i \geqslant \nu_{k_1} + \cdots + \nu_{k_{m+1}} = \mu_{m+1}$ and $\mu_m + 1 + \nu_{k_{m+1}} > \mu_m + \nu_{k_{m+1}} = \mu_{m+1}$, so $o(H_m) > \mu_{m+1}$. Our hypothesis on g implies that

$Q_m(X)$ has coefficients in $A[\{y_{sj}\}_{j=0}^{\mu_{m+1}}, \zeta_t, \ldots, \zeta_{k_{m+1}-1}]$ with total degree one in the y_{sj} and total degree $s - n + m + 1$ in the ζ_i. But $s - n + m + 1 \leq s \leq \mu_n$ and the proof of (2.6) is complete.

By assumption $\nu_{k_n+1} > (k_n + 1)[1 + \sum_{i=t}^{k_n} \nu_i] \geq \mu_n$ so in determining the coefficient of X^{μ_n} in $a_s f_s$ it suffices to consider $(a_s)_{(\mu_n)} f_{(k_n)}^s = P_n(X)$; that is, ϕ_n is the coefficient of X^{μ_n} in $P_n(X)$. If $s < n$ and $n - s \leq m < n$ or if $s \geq n$ and $1 \leq m < n$ then let ϕ_m denote the coefficient of X^{μ_m} in $P_m(X)$. It is an immediate consequence of (2.6) that $\phi_{m+1} = \phi_m \zeta_{k_{m+1}} + \psi_m$ where $\phi_m \zeta_{k_{m+1}}$ is the coefficient of $X^{\mu_{m+1}} = X^{\mu_m + \nu_{k_{m+1}}}$ in $P_m(X)\zeta_{k_{m+1}} X^{\nu_{k_{m+1}}}$ and ψ_m is the coefficient of $X^{\mu_{m+1}}$ in $Q_m(X)$. Thus, (1) of Lemma 2.5 is satisfied by the polynomials ϕ_{m+1}, ϕ_m, and ψ_m. If $m < n$ then $P_m(X)$ is clearly independent of the choice of $k_{m+1}, \ldots, k_n$ and, hence, so is ϕ_m. It is straightforward to see that $\phi_m \in D[\{y_{sj}\}_{j=0}^{\mu_m}, \zeta_t, \ldots, \zeta_{k_m}]$ so (2) of Lemma 2.5 holds. That (3) holds for the polynomials ψ_m is immediate from (2.6).

If $s < n$ we have determined polynomials $\{\phi_i\}_{i=n-s}^{n}$ and $\{\psi_i\}_{i=n-s}^{n-1}$ while for $s \geq n$ we have determined polynomials $\{\phi_i\}_{i=1}^{n}$ and $\{\psi_i\}_{i=1}^{n-1}$ that satisfy (1)–(3) of Lemma 2.5. If $s < n$ and $m = n - s$ then, by assumption, ϕ_m is the coefficient of X^{μ_m} in $P_m(X) = s!(a_s)_{(\mu_m)}$. Thus, we have $\phi_{n-s} = s! y_{s\mu_m}$. We take $\psi_{n-s-1} = \phi_{n-s}$, $\phi_i = 0$ for $0 \leq i < n - s$ and $\psi_i = 0$ for $0 \leq i < n - s - 1$. If $s \geq n$ then, by assumption, ϕ_1 is the coefficient of X^{μ_1} in

$$P_1(X) = (a_s)_{(\mu_1)}\left[(s!/(s - n + 1)!) f_{(k_1)}^{s-n+1}\right].$$

By (2.4) we can write $f_{(k_1)}^{s-n+1} = f_{(k_1-1)}^{s-n+1} + (s - n + 1)\zeta_{k_1} X^{\nu_{k_1}} f_{(k_1-1)}^{s-n} + h$ where $o(h) \geq 2\nu_{k_1} > \mu_1$. If $s > n$ then $o(X^{\nu_{k_1}} f_{(k_1-1)}^{s-n}) \geq \nu_{k_1} + 1 > \nu_{k_1} = \mu_1$ so ϕ_1 is the coefficient of X^{μ_1} in $(a_s)_{(\mu_1)}[(s!/(s - n + 1)!) f_{(k_1-1)}^{s-n+1}]$. Thus, we set $\psi_0 = \phi_1$ and $\phi_0 = 0$ and it is straightforward to see that (3) of Lemma 2.5 holds for ψ_0. Finally, if $s = n$ then ϕ_1 is the coefficient of X^{μ_1} in $P_1(X) = (a_n)_{(\mu_1)} n! f_{(k_1)} = (a_n)_{(\mu_1)} n! [f_{(k_1-1)} + \zeta_{k_1} X^{\nu_{k_1}}]$ so we can write $\phi_1 = n! y_{n,0} \zeta_{k_1} + \psi_0$ where ψ_0 is the coefficient of X^{μ_1} in $(a_n)_{(\mu_1)} n! f_{(k_1-1)}$. In particular, $\psi_0 \in A[\{y_{nj}\}_{j=0}^{\mu_1}, \zeta_t, \ldots, \zeta_{k_1-1}]$ with total degree one in the y_{nj} and total degree not greater than μ_n in the ζ_i. We set $\phi_0 = n! y_{n,0}$ and the proof of Lemma 2.5 is complete.

Let $\{\zeta_i\}_{i=1}^{\infty}$ be a subset of B and for each positive integer n set $\mathfrak{S}_n = \{g(X) = \sum_{i=t}^{\infty} \alpha_i X^{\lambda_i} | o(g) \geq n$ and for each $i \geq t$, $\alpha_i \in A[\zeta_1, \ldots, \zeta_i]$ with total degree n in $\zeta_1, \ldots, \zeta_i\}$. If $g = \sum_{i=t}^{\infty} \alpha_i X^{\lambda_i} \in \mathfrak{S}_{n_1}$ and $h = \sum_{i=s}^{\infty} \beta_i X^{\gamma_i} \in \mathfrak{S}_{n_2}$ we say that h *dominates* g provided $n_1 \leq n_2$ and $\gamma_i \geq \sum_{j=t}^{i} \lambda_j$ for $i \geq \max\{s, t\}$. Set $\mathfrak{S} = \bigcup_{n=1}^{\infty} \mathfrak{S}_n$.

COROLLARY 2.7. *Let $f(X) = \sum_{i=t}^{\infty} \zeta_i X^{\nu_i}$ where $\nu_t > t \geq 1$ and $\nu_k > k(1 + \sum_{i=t}^{k-1} \nu_i)$ for $k > t$. Assume that $G(Y) = \sum_{i=0}^{\infty} b_i Y^i$ is a nonzero element of $A[[X, \{q_\lambda\}_{\lambda \in \Gamma}]][[Y]]$, where $\{q_\lambda\}_{\lambda \in \Gamma}$ is a countable subset of $\mathfrak{S}$ such that each q_λ dominates f and such that $\{q_\lambda | o(q_\lambda) \leq k\}$ is finite for each positive integer k. Choose n such that $o(b_n) = \min\{o(b_i)\}_{i=1}^{\infty}$ and if A has nonzero characteristic p assume that we can choose $n < p$. If $G(f) = 0$ then there exists a positive integer $k_0 > t$ and a nonzero element d in $A[\zeta_1, \ldots, \zeta_{k_0}]$ such that for $k > k_0$, $d\zeta_k$ is a polynomial in $A[\zeta_1, \ldots, \zeta_{k-1}]$ with total degree not more than ν_{k+n} in $\zeta_1, \ldots, \zeta_{k-1}$.*

PROOF. Let $N = o(b_n) = \min\{o(b_i)\}_{i=1}^{\infty}$ where, by assumption, $n < p$ if $p > 1$. Since $b_0 = -\Sigma_{i=1}^{\infty} b_i f^i$ and $o(f) \geqslant 1$ we have $o(b_0) > N$. Thus, for each $i \geqslant 0$ we can write $b_i = \Sigma_{j=0}^{\infty} y_{ij} X^{j+N}$ where $y_{ij} \in B$ and $y_{n,0} \neq 0$. If we set $a_i = b_i / X^N = \Sigma_{j=0}^{\infty} y_{ij} X^j$ and $H(Y) = G(Y)/X^N$, then $H(Y) = \Sigma_{i=0}^{\infty} a_i Y^i$ and $H(f) = 0$. We apply Lemma 2.5 to $H(f)$.

Let $\mathbb{S}$ denote the set of all sequences $\{k_i\}_{i=1}^{n}$ (where n is as chosen above) of positive integers such that $\max\{N, t\} < k_1 < \cdots < k_n$. For $\sigma \in \mathbb{S}$ we denote by $\{{}_\sigma\phi_i\}_{i=0}^{n}$ and $\{{}_\sigma\psi_i\}_{i=0}^{n-1}$ the collection of polynomials obtained as in Lemma 2.5 by considering the coefficient of X^{μ_n} in $H(f)$, where $\mu_n = \nu_{k_1} + \cdots + \nu_{k_n}$. This coefficient is necessarily zero so for each $\sigma \in \mathbb{S}$ we have ${}_\sigma\phi_n = 0$. Also, for each $\sigma \in \mathbb{S}$, ${}_\sigma\phi_0 = n! y_{n,0}$ by (4) of Lemma 2.5. Since $n < p$ if $p > 1$, ${}_\sigma\phi_0 \neq 0$. Therefore there exists a largest integer m, $0 \leqslant m < n$, for which there is a sequence $\tau = \{k_i^0\}_{i=1}^{n}$ in $\mathbb{S}$ with ${}_\tau\phi_m \neq 0$. By (2) of Lemma 2.5, ${}_\tau\phi_m$ is independent of the choice of $k_{m+1}^0, \ldots, k_n^0$; that is, if $\sigma = \{k_1^0, \ldots, k_m^0, k_{m+1}, \ldots, k_n\} \in \mathbb{S}$ then ${}_\sigma\phi_m = {}_\tau\phi_m \neq 0$. Moreover, by choice of m, ${}_\sigma\phi_i = 0$ for $m + 1 \leqslant i \leqslant n$. If k is an integer such that $k > k_m^0$ then we may choose $\sigma = \{k_i\}_{i=1}^{n}$ in $\mathbb{S}$ such that $k_i = k_i^0$ for $1 \leqslant i \leqslant m$ and $k_{m+i} = k + i - 1$ for $1 \leqslant i \leqslant n - m$. For $1 \leqslant j \leqslant n$ set $\mu_j = \nu_{k_1} + \cdots + \nu_{k_j}$. By Lemma 2.5 we have

$$0 = {}_\sigma\phi_{m+1} = {}_\sigma\phi_m \zeta_{k_{m+1}} + {}_\sigma\psi_m = {}_\sigma\phi_m \zeta_k + {}_\sigma\psi_m, \ {}_\sigma\phi_m \in A\left[\{y_{ij} | 0 \leqslant i \leqslant \mu_n, 0 \leqslant j \leqslant \mu_m\}, \zeta_t, \ldots, \zeta_{k_m}\right],$$

and

$${}_\sigma\psi_m \in A\left[\{y_{ij} | 0 \leqslant i \leqslant \mu_n, 0 \leqslant j \leqslant \mu_{m+1}\}, \zeta_t, \ldots, \zeta_{k-1}\right]$$

with total degree one in the y_{ij} and total degree not greater than μ_n in the ζ_i.

For each $i \geqslant 0$, $b_i \in A[[X, \{q_\lambda\}_{\lambda \in \Gamma}]]$ so we can write $b_i = \Sigma_{j=0}^{\infty} y_{ij} X^{j+N} = h_i(X, \{q_\lambda\}_{\lambda \in \Gamma})$. For $1 \leqslant s \leqslant n$ and $0 \leqslant j \leqslant \mu_s$ we wish to further describe y_{ij}. Suppose that $q_\lambda = \Sigma_{i=t_\lambda}^{\infty} \beta_{\lambda i} X^{\gamma_{\lambda, i}}$ for each $\lambda \in \Gamma$. If $i \geqslant \max\{t_\lambda, k_s\}$ then, since q_λ dominates f, we have $\gamma_{\lambda i} \geqslant \Sigma_{j=t}^{i} \nu_j \geqslant \Sigma_{j=t}^{k_s} \nu_i \geqslant \nu_{k_1 - 1} + \nu_{k_1} + \cdots + \nu_{k_s} = \nu_{k_1 - 1} + \mu_s > (k_1 - 1)(1 + \Sigma_{j=t}^{k_1 - 2} \nu_j) + \mu_s \geqslant N + \mu_s$ since by choice $k_1 > N$. Therefore, in computing the coefficients y_{ij} of X^{j+N} for $0 \leqslant j \leqslant \mu_s$, we can replace q_λ with $(q_\lambda)_{(\gamma_{\lambda, k_{s-1}})} = \Sigma_{i=t_\lambda}^{k_s - 1} \beta_{\lambda i} X^{\gamma_{\lambda, i}}$. But $q_\lambda \in \mathfrak{S}$ so for $i \geqslant t_\lambda$ we have $\beta_{\lambda i} \in A[\zeta_1, \ldots, \zeta_i]$; therefore, $y_{ij} \in A[\zeta_1, \ldots, \zeta_{k_s - 1}]$ for $0 \leqslant j \leqslant \mu_s$. But this implies that ${}_\sigma\phi_m \in A[\zeta_1, \ldots, \zeta_{k_m}]$ and ${}_\sigma\psi_m \in A[\zeta_1, \ldots, \zeta_{k-1}]$. For $0 \leqslant j \leqslant \mu_s$ we now determine an upper bound on the degree of y_{ij} as a polynomial in the ζ_i. In $b_i = \Sigma_{j=0}^{\infty} y_{ij} X^{j+N} = h_i(X, \{q_\lambda\}_{\lambda \in \Gamma})$ suppose we consider a monomial $q_{\lambda_1}^{m_1} \cdots q_{\lambda_r}^{m_r}$. If $q_{\lambda_i} \in \mathfrak{S}_{n_i}$ then $o(q_{\lambda_i}) \geqslant n_i$ so $o(q_{\lambda_1}^{m_1} \cdots q_{\lambda_r}^{m_r}) \geqslant \Sigma_{i=1}^{r} n_i m_i$. Clearly then, only those monomials such that $\Sigma_{i=1}^{r} n_i m_i \leqslant \mu_s + N$ can contribute to y_{ij} for $0 \leqslant j \leqslant \mu_s$. Since the coefficients of q_{λ_i} have total degree n_i in the ζ_j, the coefficients of the monomial $q_{\lambda_1}^{m_1} \cdots q_{\lambda_r}^{m_r}$ have total degree $\Sigma_{i=1}^{r} n_i m_i \leqslant \mu_s + N$ in the ζ_j. Consequently, y_{ij} has total degree at most $\mu_s + N$ in $\zeta_1, \ldots, \zeta_{k_s - 1}$ so ${}_\sigma\psi_m$ has total degree at most $\mu_n + \mu_{m+1} + N \leqslant 2\mu_n + N$ in $\zeta_1, \ldots, \zeta_{k-1}$. Since $k_n = k + n - m - 1 < k + n$ we have $\nu_{k+n} > (k + n)(1 + \Sigma_{i=t}^{k+n-1} \nu_i) \geqslant N + 2\Sigma_{i=t}^{k+n-1} \nu_i \geqslant N + 2\mu_n$. If we take $d = {}_\sigma\phi_m$ and $k_0 = k_m$ the proof of Corollary 2.7 is complete.

Let $\{\xi_i\}_{i=1}^{\infty}$ be a subset of B, let $\{\nu_{ij} | i \geqslant 1, j \geqslant i\}$ and $\{\gamma_i\}_{i=1}^{\infty}$ denote the sets of positive integers as described preceding the statement of Theorem 2.1, and let $\mathfrak{F} = \{f_i\}_{i=1}^{\infty}$ be the subset of $B[[X]]$ defined by $f_i = \sum_{j=i}^{\infty} \xi_j X^{\nu_{ij}}$. By assumption there exists a nonzero power series $G(Y)$ in $A[[X]][[Y_\omega]]$ such that $G(\mathfrak{F}) = 0$. By Corollary 2.3 there exist integers $m \geqslant 0$ and $t \geqslant 1$ and a nonzero element $H(Y)$ in $A[[X, \{f_i^{p^j} | j = m \text{ and } i > t \text{ or } p > 1 \text{ and } j > m\}]][[Y]]$ such that $H(f_t^{p^m}) = 0$. Further, if $p > 1$ then $H(Y)$ is a polynomial with degree less than p. Define $f = f_t^{p^m} = \sum_{j=t}^{\infty} \zeta_j X^{\nu_j}$ where $\zeta_j = \xi_j^{p^m}$ and $\nu_j = p^m \nu_{tj}$. Then $\nu_t = p^m \nu_{tt} > t$ and for $k > t$, $\nu_k = p^m \nu_{tk} \geqslant p^m \nu_{1k} > p^m k(1 + \sum_{j=t}^{k-1} \nu_{tj}) \geqslant k(1 + \sum_{j=t}^{k-1} \nu_j)$. If $j \geqslant m$, with $j = m + r$, then set $f_s^{p^j} = g_{sr} = \sum_{i=s}^{\infty} \zeta_i^{p^r} X^{p^{m+r}\nu_{si}}$. In terms of the notation preceding Corollary 2.7 we have $g_{sr} \in \mathcal{E}_{p^r}$. If $r = 0$ then $j = m$ and $s > t$ so $p^m \nu_{si} = p^m \sum_{\lambda=s-1}^{i} \nu_{s-1,\lambda} \geqslant \sum_{\lambda=t}^{i} p^m \nu_{t\lambda} = \sum_{\lambda=t}^{i} \nu_\lambda$ and hence $g_{s,0}$ dominates f for $s > t$. If $r > 0$ then $j > m$ and $p > 1$ so $p^r \nu_{si} \geqslant 2\nu_{1,i} > \nu_{1,i} + i(1 + \sum_{\lambda<i,\delta<\lambda} \nu_{\delta\lambda}) \geqslant \nu_{1,i} + \sum_{\delta=1}^{i-1} \sum_{\lambda=\delta}^{i-1} \nu_{\delta\lambda}$. But if $\delta > 1$ then $\nu_{\delta i} = \sum_{\lambda=\delta-1}^{i} \nu_{\delta-1,i}$ so if $1 \leqslant \rho \leqslant i$ the last expression of the previous sentence equals $\nu_{\rho i} + \sum_{\delta=\rho}^{i-1} \sum_{\lambda=\delta}^{i-1} \nu_{\delta\lambda} \geqslant \sum_{\lambda=\rho}^{i} \nu_{\rho\lambda}$. In particular, if $i \geqslant t$ then, with $\rho = t$, we get that $p^{m+r} \nu_{si} > p^m \sum_{\lambda=t}^{i} \nu_{t\lambda} = \sum_{\lambda=t}^{i} \nu_\lambda$. Hence, g_{sr} dominates f when $r > 0$. By Corollary 2.7 there exist integers n and k_0 and a nonzero element d in $A[\zeta_1, \ldots, \zeta_{k_0}]$ such that for $k > k_0$, $d\zeta_k$ is a polynomial in $A[\zeta_1, \ldots, \zeta_{k-1}]$ with total degree not more than ν_{k+n} in $\zeta_1, \ldots, \zeta_{k-1}$. But $\nu_{k+n} = p^m \nu_{t,k+n} \leqslant p^m \nu_{k+n,k+n} = p^m \gamma_{k+n}$ and $\zeta_i = \xi_i^{p^m}$ so we have proved the following.

(2.8) *For each subset* $\{\xi_i\}_{i=1}^{\infty}$ *of* B *there exist integers* $n \geqslant 1$, $k_0 \geqslant 1$, *and* $m \geqslant 0$ *and a nonzero element* d *in* $A[\xi_1^{p^m}, \ldots, \xi_{k_0}^{p^m}]$ *such that for* $k > k_0$, $d\xi_k^{p^m}$ *is a polynomial in* $A[\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}]$ *with total degree not more than* $p^m \gamma_{k+n}$ *in* $\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}$.

Recall that $A \subseteq D \subseteq J$, $J = D[B]$, D and J have quotient fields K and L, respectively, and K_0 is the maximal, separable extension of K in L.

(2.9) *L is algebraic over K, L/K_0 is purely inseparable with finite exponent, and* $[K_0 : K]$ *is finite.*

Proof. To show that L/K is algebraic, it suffices to show that J is algebraic over D. But since $J = D[B]$ and $A \subseteq D$, it suffices to show that B is algebraic over A. Thus, let $\zeta \in B$ and, with $\{\gamma_i\}_{i=1}^{\infty}$ as defined in the remarks preceding the statement of Theorem 2.1, let $\xi_k = \zeta^{\gamma_{2k}^k}$ for each integer $k \geqslant 1$. By (2.8) there exist integers $n \geqslant 1$, $k_0 \geqslant 1$, $m \geqslant 0$ and a nonzero element d in $A[\xi_1^{p^m}, \ldots, \xi_{k_0}^{p^m}]$ such that, for $k > k_0$, $d\xi_k^{p^m}$ is a polynomial in $A[\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}]$ with total degree not more than $p^m \gamma_{k+n}$ in $\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}$. Since $\xi_{k-1}^{p^m} = \zeta^{p^m \gamma_{2k-2}^{k-1}}$ it follows that $d\xi_k^{p^m} = d\zeta^{p^m \gamma_{2k}^k}$ has total degree at most $p^{2m} \gamma_{k+n} \gamma_{2k-2}^{k-1}$ in ζ and if we choose $k > \max\{n, p^m\}$ then $\gamma_{2k} = \nu_{2k,2k} \geqslant \nu_{1,2k} > k\nu_{k+n,k+n} = k\gamma_{k+n} > p^m \gamma_{k+n}$. Thus, $p^{2m} \gamma_{k+n} \gamma_{2k-2}^{k-1} < p^m \gamma_{2k} \gamma_{2k}^{k-1} = p^m \gamma_{2k}^k$. Since d is a nonzero polynomial in ζ it follows that ζ is algebraic over A.

We now show the existence of a finite extension K_1 of K such that $K_1 \subseteq L$ and L/K_1 is purely inseparable with finite exponent. If $m \geqslant 0$ is an integer then $L^{p^m} \subseteq K_1$ if and only if $J^{p^m} \subseteq K_1$ and since $J = D[B]$, $J^{p^m} \subseteq K_1$ if and only if $B^{p^m} \subseteq K_1$. Thus, if the desired extension K_1 does not exist we can choose a subset

$\{\xi_i\}_{i=1}^{\infty}$ of B such that $\xi_k^{p^k}$ is not in the quotient field of $D[\xi_1, \ldots, \xi_{k-1}]$. But by (2.8) there exist integers $n \geqslant 1$, $k_0 \geqslant 1$, and $m \geqslant 0$ and a nonzero element d in $A[\xi_1^{p^m}, \ldots, \xi_{k_0}^{p^m}]$ such that, for $k > k_0$, $d\xi_k^{p^m}$ is a polynomial in $A[\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}] \subseteq D[\xi_1, \ldots, \xi_{k-1}]$. In particular, $\xi_k^{p^m}$ is in the quotient field of $D[\xi_1, \ldots, \xi_{k-1}]$ for $k \geqslant \max\{m, k_0\}$ contrary to the choice of the set $\{\xi_i\}_{i=1}^{\infty}$, so we conclude that the desired extension K_1 of K exists. Let K_0 be the maximal separable extension of K in K_1. Then $[K_0 : K]$ is finite and since $[K_1 : K_0]$ is finite, K_1/K_0 has finite exponent. It follows that L/K_0 is purely inseparable with finite exponent and hence K_0 is the maximal separable extension of K in L. This completes the proof of (2.9).

Now let D_0 be any integral domain with quotient field K_1 containing K_0 and such that $D \subseteq D_0 \subseteq J$. Then L/K_1 is purely inseparable with finite exponent m so $B^{p^m} \subseteq J^{p^m} \subseteq K_1$. Let $\{\xi_i\}_{i=1}^{\infty}$ be a subset of B and choose integers $n \geqslant 1$, $k_0 \geqslant 1$, $m_1 \geqslant 0$ and a nonzero element d_1 in $A[\xi_1^{p^{m_1}}, \ldots, \xi_{k_0}^{p^{m_1}}]$ as in (2.8) so that $d_1\xi_k^{p^{m_1}}$ is a polynomial in $A[\xi_1^{p^{m_1}}, \ldots, \xi_{k-1}^{p^{m_1}}] \subseteq D[\xi_1^{p^{m_1}}, \ldots, \xi_{k-1}^{p^{m_1}}]$ with total degree not more than $p^{m_1}\gamma_{k+n}$ in the $\xi_i^{p^{m_1}}$. If $m_1 < m$ then $(d_1\xi_k^{p^{m_1}})^{p^{m-m_1}} = d_1^{p^{m-m_1}}\xi_k^{p^m}$ is a polynomial in $D[\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}]$ with total degree not more than $p^m\gamma_{k+n}$ in the $\xi_i^{p^m}$, so we may assume that $m_1 \geqslant m$. Thus, $L^{p^{m_1}} \subseteq K_1$ and, in particular, $d_1 \in K_1$. There exists a nonzero element $d_2 \in D_0$ such that $d_2d_1 \in D_0$ and $d_2d_1\xi_i^{p^{m_1}} \in D_0$ for $1 \leqslant i \leqslant k_0$. It follows that $d_2d_1\xi_k^{p^{m_1}} \in D_0[\xi_1^{p^{m_1}}, \ldots, \xi_{k-1}^{p^{m_1}}]$ for each $k > 1$. Since D_0 is algebraic over D, $d_2d_1D_0 \cap D \neq (0)$. If d is any nonzero element of $d_2d_1D_0 \cap D$ then d meets the requirements stated in Theorem 2.1. This completes the proof of Theorem 2.1.

Corollary 2.10. *Let D_0 be as in Theorem* 2.1 *and for each positive integer k let h_k denote the polynomial $1 + x + \cdots + x^{k-1}$. For each subset $\{\xi_i\}_{i=1}^{\infty}$ of B there exists a nonzero element d in D and nonnegative integers m and n such that $d^{h_k(p^m\gamma_{k+n})}\xi_k^{p^m} \in D_0$ for each k.*

Proof. Applying Theorem 2.1 we get nonnegative integers m and n and a nonzero element d in D such that $d\xi_k^{p^m}$ is a polynomial in $D_0[\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}]$ with degree at most $p^m\gamma_{k+n}$ in the ξ_i. In particular, $d\xi_1^{p^m} = d^{h_1(p^m\gamma_{1+n})}\xi_1^{p^m} \in D_0$. Suppose we have shown that $d^{h_t(p^m\gamma_{t+n})}\xi_t^{p^m} \in D_0$ for $1 \leqslant t \leqslant k-1$. Since $h_{k-1}(p^m\gamma_{k-1+n}) \geqslant h_t(p^m\gamma_{t+n})$ for $1 \leqslant t \leqslant k-1$ we have $d^{h_{k-1}(p^m\gamma_{k-1+n})}\xi_t^{p^m} \in D_0$. It follows that $[d^{h_{k-1}(p^m\gamma_{k-1+n})}]^{p^m\gamma_{k+n}}d\xi_k^{p^m} \in D_0$. Since $\gamma_{k+n} > \gamma_{k-1+n}$ we have $h_k(p^m\gamma_{k+n}) > p^m\gamma_{k+n}h_{k-1}(p_{k-1+n}^m) + 1$, so $d^{h_k(p^m\gamma_{k+n})}\xi_k^{p^m} \in D_0$.

Corollary 2.11. *If D_0 is as in Theorem* 2.1 *then for each subset $\{\xi_i\}_{i=1}^{\infty}$ of B there exists a nonzero element d in D and a sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that $(d\xi_i)^{n_i} \in D_0$ for each i.*

Proof. Applying Corollary 2.10 to the set $\{\xi_k^{h_k(p^k\gamma_{2k})}\}_{k=1}^{\infty}$ we get nonnegative integers m and n and a nonzero element d_1 in D such that $d_1^{h_k(p^m\gamma_{k+n})}\xi_k^{p^m h_k(p^k\gamma_{2k})} \in D_0$ for each k. In particular, if $k \geqslant \max\{m, n\}$ then $(d_1\xi_k)^{p^m h_k(p^k\gamma_{2k})} \in D_0$. If d_2 is any nonzero element of D such that $d_2\xi_t^{p^m h_t(p^t\gamma_{2t})} \in D_0$ for $1 \leqslant t < \max\{m, n\}$ then Corollary 2.11 follows with $d = d_1d_2$ and $n_k = p^m h_k(p^k\gamma_{2k})$.

In some of our later results we shall wish to choose D_0 to meet certain conditions. Our next remark will allow us to do this.

REMARK 2.12. *Let D and J_0 be integral domains with quotient fields K and K_0, respectively, and suppose that K_0/K is a finite separable extension. Then there exists $\theta \in K_0$ such that $K_0 = K(\theta)$ and $D_0 = D[\theta] \subseteq J_0$ is an integral extension of D. Further, if D is integrally closed and D' is the integral closure of D in K_0, there exists a nonzero element d in D such that $dD' \subseteq D_0$.* (Cf. the proof of Theorem 41.7 in **[4]**.)

3. Domains in which (1)–(5) are equivalent. To show that (1)–(5) of (1.2) are equivalent for the integral domains A and B, where $A \subset B$, it suffices, in view of Proposition 3.7, to show that (5) → (1). Our basic approach is to reduce to the case in which B is an overring of A (that is, A and B have the same quotient field) and apply the results of **[1]**, as restated below, to show that (a) of (1.1) holds. In Lemma 3.4 it becomes necessary to pass to an overring D of A and show that if (e) holds for $A \subset B$ then (a) holds for $D \subset J$, where $J = D[B]$. Thus, as in Theorem 2.1, some of our preliminary results are stated in this more general setting. The following observation will be useful.

REMARK 3.1. *Let A, B, D, and J be integral domains such that $A \subset B \subseteq J$, $A \subseteq D \subseteq J$ and $J = D[B]$. In order to show that the domains D and J satisfy* (1) *of* (1.2) [*respectively*, (a) *of* (1.1)] *it suffices to show that* (1) [*respectively*, (a)] *holds for each subset $\{\xi_i\}_{i=1}^{\infty}$ of B.*

Our next several results, (3.2)–(3.6), will facilitate the reduction to the case in which B is an overring of A. In this case several instances are given in **[1]** in which conditions (a)–(d) of (1.1) are equivalent. The proofs given in **[1]**, with slight alterations (particularly the application of Remark 3.1 and the substitution of Theorem 2.1, Corollary 2.10, and Corollary 2.11 with $D_0 = D$, respectively, for Theorem 1.3, Lemma 1.4 and Lemma 1.5 of **[1]**), actually yield the more general results stated in Lemmas 3.2 and 3.3 below.

LEMMA 3.2. *Let A, B, D, and J be integral domains with quotient field K such that $A \subset B \subseteq J$, $A \subseteq D \subseteq J$, and $J = D[B]$. One of the following holds:*

(i) *Condition* (e) *of* (1.1) *fails for the domains A and B.*

(ii) *Condition* (a) *of* (1.1) *holds for the domains D and J.*

(iii) *For each countable subset S of nonzero elements of D there exists a subset $\{\xi_i\}_{i=1}^{\infty}$ of B such that if d, m, n are as in Theorem* 2.1 (*with $D_0 = D$*) *then d divides no element of S.*

PROOF. Cf. Proposition 2.3 of [**1**, p. 186].

LEMMA 3.3. *With notation as in Lemma* 3.2, *in each of the following cases if* (e) *of* (1.1) *holds for the domains A and B then* (a) *of* (1.1) *holds for the domains D and J.*

(1) *D is root closed in J. In this case we can take $m = 0$ in* (a) (*cf.* [**1**, *Theorem* 1.6]).

(2) *The quotient field K of D is countably generated over D* (*cf.* [**1**, *Theorem* 2.4]).

(3) *D is Noetherian* (*cf.* [**1**, *Theorem* 2.5]).

LEMMA 3.4. *Let A be an integral domain with quotient field K. If A has Krull dimension one then* (a)–(e) *of* (1.1) *are equivalent for each overring B of A.*

PROOF. Suppose that (e) holds and let $\{\xi_i\}_{i=1}^{\infty}$ be a subset of B. By Corollary 2.10 (with $A = D = D_0$) there exist a nonzero element d in A, an integer $m \geqslant 0$, and positive integers $\{n_i\}_{i=1}^{\infty}$ such that $d^{n_i}\xi_i^{p^m} \in A$ for each i. Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be the set of maximal ideals of A such that $\{\xi_i^{p^m}\}_{i=1}^{\infty} \not\subseteq A_{M_\alpha}$. If no such maximal ideals exist then $\{\xi_i^{p^m}\}_{i=1}^{\infty} \subseteq A$ and (a) holds. If $\{M_\beta\}_{\beta\in\Gamma}$ is the set of maximal ideals of A that contain d then $\{M_\beta\}_{\beta\in\Gamma} \supseteq \{M_\alpha\}_{\alpha\in\Lambda}$. In partiuclar, if $M \notin \{M_\beta\}_{\beta\in\Gamma}$ then $\{\xi_i^{p^m}\}_{i=1}^{\infty} \subseteq A_M$. Set $S = A \setminus \bigcup_{\beta\in\Gamma} M_\beta$ and let $D = A_S$. If M is a maximal ideal of A such that $M \notin \{M_\beta\}_{\beta\in\Gamma}$ then $A = M + dA$ so there exist $m \in M$ and $a \in A$ such that $1 = m + ad$. Thus, $m = 1 - ad \in M \cap S$ and $MD = D$. It follows that $\{M_\beta D\}_{\beta\in\Gamma}$ is the set of maximal ideals of D. But $d \in \bigcap_{\beta\in\Gamma} M_\beta D$ and D has Krull dimension one, so $K = D[1/d]$. By Lemma 3.3(2), (a) of (1.1) holds for the domains D and $J = D[B]$. In particular, there exist a nonzero element d_1 in D and a nonnegative integer n such that $\{d_1\xi_i^{p^n}\} \subseteq D$. If $d_2 \in d_1D \cap A$ and $k \geqslant \max\{m, n\}$ then $\{d_2\xi_i^{p^k}\} \subseteq A$ so (a) holds for the domains A and B.

REMARK 3.5. *Let $D \subseteq D_1 \subseteq J_1 \subseteq J$ be integral domains such that J_1 is an overring of D_1 with quotient field containing K_0. If one of the conditions* (1)–(5) *of* (1.2) *holds for the domains D and J then the analogous condition from* (a)–(e) *of* (1.1) *holds for the domains D_1 and J_1. Further, we can take d in D^* in* (a) *and in* (b) *we can write* $(D_1[[X]])_{D_1^*} = (D_1[[X]])_{D^*}$.

PROOF. We show that (2) → (b), the other implications being obvious. Thus, let $D_0 = D[\theta]$ be as in (2). Then $(J_1[[X]])^{p^m} \subseteq (D_0[[X]])_{D^*} \subseteq (D_1[\theta][[X]])_{D^*}$. But θ is in the quotient field of D_1 and is integral over D_1. Since D_1 is algebraic over D there exists $d \in D$ such that $dD_1[\theta] \subseteq D_1$. It follows that $(J_1[[X]])^{p^m} \subseteq (D_1[[X]])_{D^*}$.

LEMMA 3.6. *Let D and J be integral domains such that $D \subseteq J$ and* (5) *of* (1.2) *holds and let $D_0 = D[\theta]$ be an integral extension of D with quotient field K_0. Set $J_0 = J \cap K_0$ and let $\bar{D}$ be the integral closure of D in J_0. The domains D and J satisfy* (1) *of* (1.2) *if and only if the domains D_0 and $\bar{D}$ satisfy* (a) *of* (1.1).

PROOF. Suppose that $D_0 \subseteq \bar{D}$ satisfies (a) of (1.1) and let $\{\xi_i\}_{i=1}^{\infty}$ be a subset of J. By Theorem 2.1, L has finite exponent over K_0 so there exists a positive integer n such that $J^{p^n} \subseteq K_0$. If we set $\zeta_i = \xi_i^{p^n}$ for each i then $\{\zeta_i\}_{i=1}^{\infty} \subseteq J_0$ and by Corollary 2.11 there exist a nonzero element d_1 in D and a sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that $(d_1\zeta_i)^{n_i} \in D_0 \subseteq \bar{D}$ for each i. Thus, $\{d_1\zeta_i\}_{i=1}^{\infty}$ is a subset of $\bar{D}$. Since (a) holds for $D_0 \subseteq \bar{D}$ there exist a nonnegative integer m and a nonzero element d_2 in D such that $d_2(d_1\zeta_i)^{p^m} \in D_0$ for each i. If $d = d_2d_1^{p^m}$ then $d\xi_i^{p^{m+n}} \in D_0$ for each i. Now let D_1 be an integral domain such that $D \subseteq D_1 \subseteq J$ and D_1 has quotient field containing K_0. Then $d\xi_i^{p^{m+n}} \in D_0 = D[\theta] \subseteq D_1[\theta]$ for each i and, as in the proof of Remark 3.5, there exists a nonzero element d_3 in D such that $d_3D_1[\theta] \subseteq D_1$. Thus, $d_3d\xi^{p^{m+n}} \in D_1$ for each i and (1) holds for D and J. The converse is given in Remark 3.5.

PROPOSITION 3.7. *The following implications hold between conditions* (1)–(5) *of* (1.2).

$$(1) \leftrightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (5).$$

PROOF. The implications (3) → (4) → (5) are evident and that (1) → (2) follows from Remark 2.12 and an adaptation of the proof of Proposition 2.1 of [**1**].

To show that (2) → (1) let $D_0 = D[\theta]$ be as in (2) and let $\bar{D}$ be as in Lemma 3.6. By Remark 3.5 the domains D_0 and $\bar{D}$ satisfy (b) of (1.1). But (b) → (a) [**1**, Proposition 2.1] so by Lemma 3.6 the domains D and J satisfy (1).

To see that (2) → (3) let $D_0 = D[\theta]$ be as in (2). By [**3**, Lemma 2.5] $D_0[[X]]$ is integral over $D[[X]]$ and, clearly, $J[[X]]$ is algebraic over $D_0[[X]]$. Thus, $J[[X]]$ is algebraic over $D_0[[X]]$.

Let K and L be fields with $K \subseteq L$. In [**3**] Gilmer has determined necessary and sufficient conditions in order that $L[[X_1, \ldots, X_n]]$ be integral over $K[[X_1, \ldots, X_n]]$. The main results of [**3**] are essentially summarized in [**3**, Corollary 4.2] and may be restated as follows.

(3.8) *$L[[X_1, \ldots, X_n]]$ is integral over $K[[X_1, \ldots, X_n]]$ if and only if L has finite exponent over K_0 and $[K_0: K]$ is finite, where K_0 is the maximal separable extension of K in L.*

Our next result, Theorem 3.9, shows that (1)–(5) of (1.2) are equivalent for Noetherian domains and extends (3.8) to Noetherian domains.

THEOREM 3.9. *If D is a Noetherian integral domain and J is an integral domain containing D then each of the following statements is equivalent to* (1)–(5) *of* (1.2).

(i) *There exists a nonnegative integer m and a finite integral extension D_1 of D such that $D \subseteq D_1 \subseteq J$, D_1 has quotient field K_0, and $J^{p^m} \subseteq D_1$.*

(ii) *If $Y_\Delta = \{y_i\}_{i \in \Delta}$ is any set of analytic indeterminates over D then $J[[Y_\Delta]]$ is integral over $D[[Y_\Delta]]$.*

(iii) *$J[[X]]$ is integral over $D[[X]]$.*

PROOF. (i) → (ii). If $J^{p^m} \subseteq D_1$ then clearly $J[[Y_\Delta]]$ is integral over $D_1[[Y_\Delta]]$. But D_1 is a finite D-module so $D_1[[Y_\Delta]]$ is a finite $D[[Y_\Delta]]$-module (cf. [**3**, Lemma 2.5]). In particular, $D_1[[Y_\Delta]]$ is integral over $D[[Y_\Delta]]$ and, hence, $J[[Y_\Delta]]$ is integral over $D[[Y_\Delta]]$.

That (ii) → (iii) is clear and (iii) → (3) → (4) → (5).

(5) → (1). By Theorem 2.1 and Remark 2.12 there exists an integral extension $D_0 = D[\theta]$ of D with quotient field K_0 such that $D \subseteq D_0 \subseteq J$. With notation as in Lemma 3.6, the domains D_0 and $\bar{D}$ satisfy (e) of (1.1). But D_0 is Noetherian so by Lemma 3.3, D_0 and $\bar{D}$ satisfy (a) of (1.1). It follows from Lemma 3.6 that the domains D and J satisfy (1) of (1.2). In particular, it follows from Proposition 3.7 that (1)–(5) are equivalent.

(2) → (i). Let D_0 and m be as in (2). Thus, D_0 has quotient field K_0 and $J^{p^m} \subseteq K_0$. If we set $D_1 = D_0[J^{p^m}]$ then $D \subseteq D_1 \subseteq J$, D_1 has quotient field K_0, and $J^{p^m} \subseteq D_1$. To show that (i) holds it suffices to show that D_1 is finitely generated as a D-module. Thus, let $\{\lambda_i\}_{i=1}^{\infty}$ be a subset of D_1 and let M be the D-module generated by $\{\lambda_i\}_{i=1}^{\infty}$. By (2) there exists a nonzero element d in D such that $\{d\lambda_i\}_{i=1}^{\infty} \subseteq D_0$. Thus, dM is a submodule of the Noetherian D-module D_0 and, hence, dM is finitely generated. It follows that M, and therefore D_1, is finitely generated as a D-module.

THEOREM 3.10. *If D is an integrally closed integral domain then* (1)–(5) *of* (1.2) *are equivalent.*

PROOF. Let J be an integral domain that contains D and suppose that (5) of (1.2) holds for D and J. It follows from Theorem 2.1 and Remark 2.12 that there exists a finite integral extension $D_0 = D[\theta]$ of D with quotient field K_0 such that $D \subseteq D_0 \subseteq J$. Further, if D' is the integral closure of D in K_0 then, as noted in Remark 2.12, there exists a nonzero element d in D such that $dD' \subseteq D_0$. If $\bar{D}$ is the integral closure of D in $J_0 = J \cap K_0$, then $\bar{D} \subseteq D'$ so $d\bar{D} \subseteq D_0$. Therefore the domains D_0 and $\bar{D}$ satisfy (a) of (1.1) and by Lemma 3.6 the domains D and J satisfy (1) of (1.2).

THEOREM 3.11. *If the quotient field K of D is countably generated over D then* (1)–(5) *of* (1.2) *are equivalent.*

PROOF. Suppose that $D \subset J$ and that (5) of (1.2) holds. If D_0 and $\bar{D}$ are as in the proof of Theorem 3.10 above then D_0 and $\bar{D}$ satisfy (e) of (1.1) by Remark 3.5. Moreover, since $K_0 = (D_0)_{D^*}$, K_0 is countably generated over D_0. By Lemma 3.3, (a) of (1.1) holds for D_0 and $\bar{D}$ so by Lemma 3.6, (1) of (1.2) holds for D and J.

THEOREM 3.12. *If D has Krull dimension one then* (1)–(5) *of* (1.2) *are equivalent.*

PROOF. Let J, D_0 and $\bar{D}$ be as in the preceding two proofs. Then D_0 and $\bar{D}$ satisfy (e) of (1.1) and D_0 has Krull dimension one. The theorem follows from Lemmas 3.4 and 3.6.

The four previous theorems have established sufficient conditions on D in order that (1)–(5) of (1.2) be equivalent. Our next theorem, Theorem 3.14, will give sufficient conditions on J. We first prove the following result.

PROPOSITION 3.13. *Let D, D_0 and J be as in Theorem* 2.1. *For each subset $\{\zeta_i\}_{i=1}^{\infty}$ of J there exists a nonzero element d in D such that $d \in \bigcap_{i=1}^{\infty} \zeta_i^{-1} D_0[\zeta_1^{-1}, \ldots, \zeta_i^{-1}]$.*

PROOF. Suppose that $J^{p^{m_1}} \subseteq K_0$ and let $\delta_i = \zeta_i^{p^{m_1}}$ for each i. Set $\xi_1 = \delta_1$ and for $k > 1$ set $\xi_k = \delta_k(\xi_1 \cdots \xi_{k-1})^{p^k \gamma_{2k}}$ where $\{\gamma_i\}_{i=1}^{\infty}$ is as in Theorem 2.1. By Theorem 2.1 there exist integers $n \geqslant 1$, $m \geqslant 0$ and a nonzero element d_1 in D such that, for each $k \geqslant 1$, $d_1 \xi_k^{p^m} = g_k(\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}) \in D_0[\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}]$ with total degree at most $p^m \gamma_{k+n}$ in $\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}$. If $k \geqslant k_0 = \max\{m, n\}$ then

$$\begin{aligned} d_1 \delta_k^{p^m} &= d_1 \xi_k^{p^m} (\xi_1 \cdots \xi_{k-1})^{-p^{k+m}\gamma_{2k}} \\ &= \left(\xi_1^{p^m} \cdots \xi_{k-1}^{p^m}\right)^{-p^k \gamma_{2k}} g_k\left(\xi_1^{p^m}, \ldots, \xi_{k-1}^{p^m}\right) \in D_0\left[\xi_1^{-p^m}, \ldots, \xi_{k-1}^{-p^m}\right]. \end{aligned}$$

Since $\{\delta_i^{p^m}\}$ is contained in the quotient field of D_0 and since $(D_0)_{D^*}$ is the quotient field of D_0, there exists a nonzero element d_2 in D such that $d_2 \delta_i^{p^m} \in D_0$ for $1 \leqslant i \leqslant k_0$. If we set $d = d_1 d_2$ then $d \in \delta_i^{-p^m} D_0[\xi_1^{-p^m}, \ldots, \xi_{i-1}^{-p^m}] \subseteq \zeta_i^{-1} D_0[\zeta_1^{-1}, \ldots, \zeta_i^{-1}]$ for each i.

THEOREM 3.14 (CF. [**1**, THEOREM 1.10]). *Let $J = (D_1)_S$ where D_1 is a finite integral extension of D and S is a multiplicative system in D_1. Then* (1)–(5) *of* (1.2) *are equivalent to the condition that for each subset $\{s_i\}_{i=1}^{\infty}$ of S there exists a nonzero element d in D such that $d \in \bigcap_{i=1}^{\infty} s_i D_1$.*

PROOF. Suppose that (5) of (1.2) holds for the domains D and $J = (D_1)_S$ and let $\{s_i\}_{i=1}^{\infty}$ be a subset of S. Since, in the notation of Theorem 2.1, L is the quotient field of D_1, the domains D, D_1, and J satisfy the hypotheses of Proposition 3.13. If we take $\zeta_i = 1/s_i$ it follows from Proposition 3.13 that there exists a nonzero element d in D such that $d \in \bigcap_{i=1}^{\infty} s_i D_1$. To see that this condition implies (1) of (1.2) let D_0 be any integral domain such that $D \subseteq D_0 \subseteq J$ and D_0 has quotient field containing K_0 and let $\{\xi_i\}_{i=1}^{\infty}$ be a subset of $J = (D_1)_S$. Choose $s_i \in S$ so that $s_i\xi_i \in D_1$ and let d be a nonzero element of D such that $d \in \bigcap_{i=1}^{\infty} s_i D_1$. Then $\{d\xi_i\}_{i=1}^{\infty} \subseteq D_1$. Suppose that $D_1 = D[\lambda_1, \ldots, \lambda_n]$ where each λ_i is integral over D. Since $[L : K]$ is finite by assumption, L has finite exponent over K_0. Thus, there exists an integer $m \geqslant 0$ such that $\{\lambda_i^{p^m}\}_{i=1}^{n} \subseteq K_0$. Therefore, $\{d^{p^m}\xi_i^{p^m}\}_{i=1}^{\infty} \subseteq D_1^{p^m} \subseteq D[\lambda_1^{p^m}, \ldots, \lambda_n^{p^m}] \subseteq D_0[\lambda_1^{p^m}, \ldots, \lambda_n^{p^m}]$. But there exists $d_1 \in D^*$ such that $d_1 D_0[\lambda_1^{p^m}, \ldots, \lambda_n^{p^m}] \subseteq D_0$ so if $d_2 = d_1 d^{p^m}$ then $\{d_2\xi_i^{p^m}\}_{i=1}^{\infty} \subseteq D_0$ and (1) holds.

PROPOSITION 3.15. *Let A, B, and C be integral domains with quotient fields F, K, and L, respectively, such that $A \subset B \subset C$. If conditions* (1)–(5) *of* (1.2) *are equivalent for the domains A and B and for the domains B and C, then they are equivalent for the domains A and C.*

PROOF. Suppose that (5) holds for the domains A and C and let F_0 be the maximal, separable extension of F in L. By Theorem 2.1 L is algebraic over F, $[F_0 : F]$ is finite, and L has finite exponent over F_0. To show that (1) holds let A_0 be an integral domain with quotient field containing F_0 such that $A \subseteq A_0 \subseteq C$ and let $\{\xi_i\}_{i=1}^{\infty}$ be a subset of C. Let K_0 be the maximal separable extension of K in L and set $B_0 = B[A_0]$. Clearly, L is purely inseparable over the quotient field of B_0 (since it contains F_0), so K_0 is contained in the quotient field of B_0. But (5) also holds for the domains B and C and (1)–(5) of (1.2) are equivalent for these domains, so there exist a nonzero element b in B and an integer $m_1 \geqslant 0$ such that $\{b\xi_i^{p^{m_1}}\} \subseteq B_0 = B[A_0]$ for each i. For each i, write $b\xi_i^{p^{m_1}} = \sum_{j=1}^{k_i} b_{ij}\alpha_{ij}$ where $b_{ij} \in B$ and $\alpha_{ij} \in A_0$. If F_1 is the maximal separable extension of F in K then $F_1 = F_0 \cap K$, so $A_0 \cap B$ has quotient field containing F_1. Since (5) $\rightarrow$ (1) for the domains A and B, there exist a nonzero element a_1 in A and an integer $m_2 \geqslant 0$ such that $\{a_1 b_{ij}^{p^{m_2}}\} \subseteq A_0 \cap B \subseteq A_0$. Thus, $a_1 b^{p^{m_2}} \xi_i^{p^{m_1+m_2}} = \sum_{j=1}^{k_i} a_1 b_{ij}^{p^{m_2}} \alpha_{ij}^{p^{m_2}} \in A_0$ for each i. If a is any nonzero element in $(a_1 b^{p^{m_2}})B \cap A$ and if $m = m_1 + m_2$ then $\{a\xi_i^{p^m}\}_{i=1}^{\infty} \subseteq A_0$ and (1) holds.

COROLLARY 3.16. *Let D and J be integral domains with quotient fields K and L, respectively, and suppose that $D \subset J$. Set $T = J \cap K$ and suppose that D is root closed in T or T is a quotient overring of D. Then in each of the following cases* (1)–(5) *are equivalent for the domains D and J.*

(1) *T is Noetherian.*

(2) *T is integrally closed.*

(3) *K is countably generated over T.*

(4) *T has Krull dimension one.*

(5) *$J = (T_1)_S$ where T_1 is a finite integral extension of T and S is a multiplicative system in T_1.*

PROOF. The result is an immediate consequence of Proposition 3.15, Lemma 3.3(1), and Theorems 3.9, 3.10, 3.11, 3.12, and 3.14.

We conclude by proving the equivalence of stronger versions of conditions (3)–(5) of (1.2).

THEOREM 3.17. *If D and J are integral domains such that $D \subset J$ then the following statements are equivalent.*

(1) *$J[[X_1, \ldots, X_n]]$ is algebraic over $D[[X_1, \ldots, X_n]]$ for each positive integer n.*

(2) *$J[[X_1, \ldots, X_n]]$ has finite transcendence degree over $D[[X_1, \ldots, X_n]]$ for each positive integer n.*

(3) *For each positive integer n, if R is an integral domain such that $D[[X_1, \ldots, X_n]] \subset R \subseteq J[[X_1, \ldots, X_n]]$ then $R \ncong D[[X_1, \ldots, X_n]][[Y_\omega]]$ via a $D[[X_1, \ldots, X_n]]$-isomorphism.*

PROOF. It suffices to show that (3) $\to$ (1). Thus if n is any positive integer and R is an integral domain such that $D[[X_1, \ldots, X_n]][[X_{n+1}]] \subset R \subseteq J[[X_1, \ldots, X_n]][[X_{n+1}]]$ then $R \ncong D[[X_1, \ldots, X_{n+1}]][[Y_\omega]]$ via a $D[[X_1, \ldots, X_{n+1}]]$-isomorphism. By Theorem 2.1, if L_1 and K_1 are the quotient fields of $J[[X_1, \ldots, X_n]]$ and $D[[X_1, \ldots, X_n]]$, respectively, then L_1 is algebraic over K_1. In particular, $J[[X_1, \ldots, X_n]]$ is algebraic over $D[[X_1, \ldots, X_n]]$.

REFERENCES

1. J. T. Arnold and D. W. Boyd, *Transcendence degree in power series rings*, J. Algebra **57** (1979), 180–195.

2. R. Gilmer, *A note on the quotient field of $D[[X]]$*, Proc. Amer. Math. Soc. **118** (1967), 1138–1140.

3. ______, *Integral dependence in power series rings*, J. Algebra **11** (1969), 488–502.

4. ______, *Multiplicative ideal theory*, Dekker, New York, 1972.

5. P. B. Sheldon, *How changing $D[[X]]$ changes its quotient field*, Trans. Amer. Math. Soc. **122** (1966), 321–333.

6. B. L. Van der Waerden, *Modern algebra*. I, Ungar, New York, 1949.

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061